

# Rate-cost tradeoffs in control

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**Abstract**—Consider a distributed control problem with a communication channel connecting the observer of a linear stochastic system to the controller. The goal of the controller is minimize a quadratic cost function. The most basic special case of that cost function is the mean-square deviation of the system state from the desired state. We study the fundamental tradeoff between the communication rate  $r$  bits/sec and the limsup of the expected cost  $b$ , and show a lower bound on the rate necessary to attain  $b$ . The bound applies as long as the system noise has a probability density function. If target cost  $b$  is not too large, that bound can be closely approached by a simple lattice quantization scheme that only quantizes the innovation, that is, the difference between the controller’s belief about the current state and the true state.

**Index Terms**—Linear stochastic control, rate-distortion tradeoff, high resolution, sequential rate-distortion theory, Shannon’s lower bound.

## I. INTRODUCTION

### A. System model

Consider the following discrete time stochastic linear system:

$$X_{t+1} = AX_t + BU_t + V_t, \quad (1)$$

where  $X_t \in \mathbb{R}^n$  is the state,  $V_t \in \mathbb{R}^n$  is the process noise,  $U_t \in \mathbb{R}^m$  are deterministic controls, and  $A$  and  $B$  are fixed matrices of dimensions  $n \times n$  and  $n \times m$ , respectively. See Fig. 1. At time  $t$ , the controller observes output  $G_t$  of the channel, and chooses a control action  $U_t$  based on everything it has observed up to time  $t$ . That is,  $U_t \in \mathcal{G}_t$ , where  $\mathcal{G}_t$  is the  $\sigma$ -algebra generated by  $G^t = (G_0, G_1, \dots, G_t)$ . At time  $t$ , the encoder observes  $X_t$  and forms a binary codeword  $F_t$ , which is then passed on to the channel. Like the controller, the encoder has access to the entire history of the data it has observed. We assume that system noises  $V_1, V_2, \dots$  are i.i.d. independent of everything else, distributed the same as  $V$ .

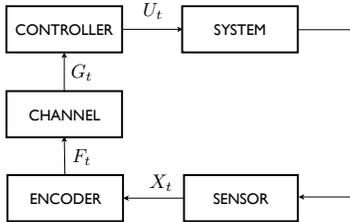


Fig. 1: The distributed control system.

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### B. The rate-cost tradeoff

The efficiency of a given control law at time  $t$  is measured by the linear quadratic regulator (LQR) cost function:

$$\frac{1}{t+1} \mathbb{E} \left[ \sum_{i=0}^t (X_i^T Q X_i + U_i^T R U_i) + X_{t+1}^T S_{t+1} X_{t+1} \right], \quad (2)$$

where  $Q \geq 0$  (positive semidefinite),  $R \geq 0$  and  $S_{t+1} \geq 0$  are matrices satisfying either of the following conditions:

- (a) Either  $R > 0$ , or
- (b)  $Q > 0$  and  $\text{rank } B = n$ .

In the special case  $Q = I_n$ ,  $R = 0$  and  $S_{t+1} = I_n$ , the cost function in (2) is the average mean-square deviation of the system from the desired state 0,

$$\frac{1}{t+1} \mathbb{E} \left[ \sum_{i=0}^{t+1} \|X_i\|^2 \right]. \quad (3)$$

More generally, the LQR cost balances between the deviation of the system from the desired state 0, and the required control power defined with respect to the norms induced by the matrices  $Q$ ,  $R$  and  $S_{t+1}$ .

For  $t \geq 0$  and  $r > 0$ , consider the following optimization problem:

$$\mathbb{B}_t(r) \triangleq \frac{1}{t+1} \min_{\substack{U_i, F_i, G_i, i=1, \dots, t-1: \\ U_i \in \mathcal{G}_i, I(F_i; G_i | G^{i-1}) \leq r}} \mathbb{E} \left[ \sum_{i=0}^t (X_i^T Q X_i + U_i^T R U_i) + X_{t+1}^T S_{t+1} X_{t+1} \right], \quad (4)$$

where  $I(F_i; G_i | G^{i-1})$  is the mutual information between  $F_i$  and  $G_i$  given the history of observed channel outputs,  $G^{i-1}$ .

The following information-theoretic quantity will play a central role in determining the operational fundamental limits of control under communication constraints.

**Definition 1** (rate-cost function). *The rate-cost function is defined as*

$$\mathbb{R}(b) \triangleq \min \left\{ r : \limsup_{t \rightarrow \infty} \mathbb{B}_t(r) < b \right\}. \quad (5)$$

In this paper, we will show a simple lower bound to the rate-cost function (5), and we will show that (5) is linked to the minimum data rate required to keep the system at LQR cost  $b$ , over both noiseless and noisy channels. Furthermore, we will demonstrate that our lower bound can be closely approached by a simple variable-length lattice-based quantization scheme that transmits only the innovation.

### C. Prior art

The analysis of control under communication constraints has a rich history. The first results on the minimum data rate required for stabilizability appeared in [3], [4]. These works analyze the evolution of a scalar plant from a worst-case perspective. In that setting, the initial condition of the plant  $X_0$  is assumed to belong to a bounded set, the process noise  $V_1, V_2, \dots$  is assumed to be bounded, and the plant is said to be *stabilizable* if there exists a (rate-constrained) control sequence such that the worst-case deviation of the system state from the target state 0 is bounded:  $\limsup_{t \rightarrow \infty} \|X_t\| < \infty$ . In [3], [4], it was shown that a fully observed unstable scalar plant

$$X_{t+1} = aX_t + U_t + V_t, \quad (6)$$

where  $a > 1$  and  $X_0, U_t$  and  $V_t$  are scalars, can be kept bounded by quantized control if and only if the available data rate exceeds  $\log a$  bits per sample. Tatikonda and Mitter [5] generalized this result to vector systems; specifically, they showed that the necessary condition for a fully observed vector plant to be stabilizable is

$$r > \sum_{i: |\lambda_i(A)| \geq 1} \log |\lambda_i(A)|, \quad (7)$$

where the sum is over the unstable eigenvalues of  $A$ , i.e. those eigenvalues whose magnitude exceeds 1. Compellingly, (7) shows that only the nonstable modes of  $A$  matter; the stable modes can be kept bounded at any arbitrarily small quantization rate (and even at zero rate if  $v_t \equiv 0$ ). In a setting without system noise, Yüksel and Başar [6] showed that variants of (7) remain necessary under three different stability criteria. Using a volume-based argument, Nair et al. [7] showed a lower bound to quantization rate in order to attain  $\limsup_{t \rightarrow \infty} \|X_t\| \leq d$ , thereby refining (7). Nair et al. [7] also presented an achievability scheme confirming that for scalar systems, that bound is tight.

Nair and Evans [8] showed that Tatikonda and Mitter's condition on the rate (7) continues to be necessary and sufficient in order to keep the mean-square deviation of the plant state from 0 bounded, that is, in order to satisfy

$$\limsup_{t \rightarrow \infty} \mathbb{E} [\|X_t\|^2] < \infty. \quad (8)$$

The power of Nair and Evans' result [8] is that unlike the results of [3]–[7], it applies to systems with unbounded process and observation disturbances. Nair and Evans' converse bound [8] applies to fixed-rate quantizers, that is, compressors whose outputs can take one of  $2^r$  values. Time-invariant fixed-rate quantizers are unable to attain bounded cost [8]. Yüksel [9], [10] considered fixed-rate quantizers with adaptive bin sizes and showed that there exists a unique invariant distribution for the system state and the quantizer parameters in [9], and studied the existence and the structure of optimal quantization and control policies in [10].

For average rate quantization, Silva et al. [11] noticed that the rate of a quantizer embedded into a feedback loop of

a control system is lower bounded by the directed mutual information from its output to the input. As discussed in [11], the bound is approached to within 1 bit by a *dithered* prefix-free quantizer, a compression setting in which both the compressor and the decompressor have access to a common dither - a random signal with special statistical properties.

Going beyond mean-square stabilizability, consider a dynamic system in (1) with quadratic cost (2), which involves an energy cost in addition to the cost on the deviation of the state from the target. In the absence of communication constraints, the minimum cost can be written as

$$b_{t \min} \triangleq \mathbb{B}_t(\infty) \quad (9)$$

The minimum quadratic cost due to the system noise attainable in the limit of infinite time can be expressed as (e.g. [12])

$$b_{\min} \triangleq \limsup_{t \rightarrow \infty} b_{t \min} = \text{tr}(\Sigma_V S), \quad (10)$$

where

- $S$  is the solution to the algebraic Riccati equation

$$S = Q + A^T (S - M) A, \quad (11)$$

$$L \triangleq (R + B^T S B)^{-1} B^T S, \quad (12)$$

$$M \triangleq L^T (R + B^T S B) L \quad (13)$$

$$= S B (R + B^T S B)^{-1} B^T S. \quad (14)$$

- $\Sigma_V$  is the covariance matrix of each of the  $V_1, V_2, \dots$ ,

$$\Sigma_V \triangleq \mathbb{E} [(V - \mathbb{E}[V])(V - \mathbb{E}[V])^T]. \quad (15)$$

The setting in which both the system noise and the observation noise are Gaussian, and the cost is the LQR cost in (2), is known as linear quadratic Gaussian (LQG) control. The optimal performance of vector quantization for the LQG control was first studied by Fischer [13], who showed that the optimal performance can be attained by first estimating and then quantizing the state, and that the minimum attainable cost decomposes into an unquantized LQG cost and a quantization cost:

$$\mathbb{B}_t(r) = b_{t \min} + \mathbb{D}_t(r), \quad (16)$$

where  $\mathbb{D}_t(r)$  is the minimum normalized accumulated distortion in sequential quantization of the state estimates. We say that a quantizer is *sequential* if its output at each time instant can depend on the entire history up to that time instant but cannot depend on the future. This causality constraint arises naturally in the dynamic control setting of Fig. 1.

The function  $\mathbb{D}_t(r)$  is challenging to evaluate exactly, and no closed-form expression is known for it. Moreover, strategies that attain (16) are in general complicated: e.g., Fu [14] gave examples illustrating that quantizers designed to minimize the distortion at current step do not generally attain the optimal performance.

In the context of a fully observed LQG system with communication constraints, Tatikonda et al. [15] noticed that  $\mathbb{D}_t(r)$  is closely tied to the *causal rate-distortion function*,

an information-theoretic quantity introduced decades earlier by Gorbunov and Pinsker [16]. Using information-theoretic tools enabled by that important observation, Tatikonda et al. [15] obtained a formula for the function  $\mathbb{R}(b)$  (defined in (5)) for a scalar Gaussian system:

$$\mathbb{R}(b) = \left| \log |a| + \frac{1}{2} \log \left( 1 + \frac{M \text{Var}[V]}{b - b_{\min}} \right) \right|_+, \quad (17)$$

where  $|\cdot|_+ \triangleq \max(0, \cdot)$ , and  $M > 0$  is the scalar given by (14). The right side of (17) is lower bounded by  $|\log |a||_+$ , thus at least  $\log |a|$  bits per time unit is necessary to attain a bounded LQR cost in an unstable ( $|a| > 1$ ) Gaussian system.

Leveraging the high resolution quantization results of Linder and Zamir [17], Tatikonda et al. [15] argued that for any time horizon  $t$ , the ratio of normalized output entropy of the best sequential quantizer,  $\frac{1}{t} H(F^t)$ , over the functional inverse of (4) converges to 1 as  $b \downarrow b_{\min, t}$ . Tatikonda et al. [15] also observed that (17) can be achieved without delay over a special Gaussian channel that is probabilistically matched to the statistics of  $V_1, V_2, \dots$  and the target cost  $b$ . An expression for  $\mathbb{R}(b)$  for vector Gaussian systems is also put forth in [15], however its proof appears to contain a gap.

For vector Gaussian systems with scalar observation and control signals, Silva et al. [18] recently showed that sequential rate-distortion function is attained by Gaussian transition probability kernels. Silva et al. [18] also computed a lower bound to the minimum rate and proposed a dithered quantization scheme that performs within 1.254 bits from it. For vector Gaussian systems, Tanaka et al. [19] proposed a semidefinite program to evaluate the Gaussian sequential rate-distortion function.

#### D. Our contribution

In this paper, we show a lower bound to  $\mathbb{R}(b)$  of a fully observed system, which holds as long as the system noise  $V_t$  is continuous. We do not require the noise to be bounded or Gaussian. If the system is scalar and the system noise is Gaussian, the new bound reduces to (17). We also show that (7) remains necessary to keep the LQR cost bounded, even if the system noise is non-Gaussian, generalizing previously known results. Although our converse lower bound holds for a general class of codes that can take full advantage of the memory of the data observed so far and that are not constrained to be linear or have any other particular structure, we show that the new bound can be closely approached within a much more narrow class of codes. Namely, a simple variable-rate quantization scheme that uses a lattice covering and that only transmits the difference between the controller's estimate about the current system state and the true state performs within a fraction of a bit from the lower bound, with a vanishing gap as  $b$  approaches  $b_{\min}$ . Unlike previously proposed strategies, our scheme does not use the dither.

#### E. Technical approach

The main idea behind our approach to show a converse (impossibility) result is to recursively lower-bound distortion-

rate functions arising at each step. We apply the classical Shannon's lower bound to distortion-rate function [20], which bounds the distortion-rate function  $X$  in terms of the entropy power of  $X$ , and we use the entropy power inequality [21], [22] to split up the distortion-rate functions of the sums of independent random variables. Since Shannon's lower bound applies as long as the source random variable has a density, our technique circumvents a precise characterization of the distribution of the state at each time instant. The technique also does not restrict the system noises to be Gaussian.

To show that our bound can be approached at high rates, we build on the ideas from high resolution quantization theory. A pioneering result of Gish and Pierce [23] states that in the limit of high resolution, a uniform scalar quantizer incurs a loss of only about  $\frac{1}{2} \log_2 \frac{2\pi e}{12} \approx 0.254$  bits per sample. Ziv [24] showed that regardless of target distortion, the normalized output entropy of a dithered scalar quantizer exceeds that of the optimal vector quantizer by at most  $\frac{1}{2} \log \frac{4\pi e}{12} \approx 0.754$  bits per sample. A lattice quantizer presents a natural extension of a scalar uniform quantizer to multiple dimensions. The advantage of lattice quantizers over uniform scalar quantizers is that the shape of their quantization cells can be made to approach a Euclidean ball in high dimensions. Indeed, relying on a fundamental result by Rogers [25] and crediting Poltyev, Zamir and Feder [26, (25)] showed that the logarithm of the normalized second moment of the best  $n$ -dimensional lattice quantizer converges to that of a ball,  $\log \frac{1}{2\pi e}$ , at a speed  $O\left(\frac{\log n}{n}\right)$ . The works of Gersho [27], Zamir and Feder [28] and Linder and Zeger [29] established that the entropy rate of dithered lattice quantizers converges to Shannon's lower bound in the limit of vanishing distortion.

While the presence of a dither signal both at the encoder and the decoder greatly simplifies the analysis and can improve the quantization performance, it also complicates the engineering implementation. In this paper, we do not consider dithered quantization. Neither do we rely directly on the classical heuristic reasoning by Gish and Pierce [23]. Instead, we use a non-dithered lattice quantizer followed by an entropy coder. To rigorously prove that its performance approaches our converse bound, we employ a recent upper bound [30], [31] on the output entropy of lattice quantizers in terms of the differential entropy of the source, the target distortion and a smoothness parameter of the source density.

#### F. Paper organization

In Section II, we state and discuss our results. Section III presents the proof of the main converse result.

## II. RESULTS

Our results are expressed in terms of the entropy power of the system and observation noises. The entropy power of

an  $n$ -dimensional random vector  $X$  is defined as<sup>1</sup>

$$N(X) \triangleq \frac{1}{2\pi e} \exp\left(\frac{2}{n}h(X)\right), \quad (18)$$

where  $h(X) = -\int_{\mathbb{R}^n} f_X(x) \log f_X(x) dx$  is the differential entropy of  $X$ , and  $f_X(\cdot)$  is the density of  $X$  with respect to the Lebesgue measure on  $\mathbb{R}^n$ .

The entropy power satisfies the following inequalities:

$$N(X) \leq (\det \Sigma_X)^{\frac{1}{n}} \leq \frac{1}{n} \text{Var}[X] \quad (19)$$

where the variance of  $X$  can be written as

$$\text{Var}[X] = \mathbb{E}[\|X - \mathbb{E}[X]\|^2] = \text{tr} \Sigma_X. \quad (20)$$

The first equality in (19) is attained if and only if  $X$  is Gaussian and the second if and only if  $X$  is white. Thus,  $N(X)$  is equal to the normalized variance of a white Gaussian random vector with differential entropy  $h(X)$ .

Our first result is a lower bound on the rate-cost function.

**Theorem 1.** *At any LQR cost  $b > b_{\min}$ , the rate-cost function is bounded below as follows.*

- If rank  $B = n$ , then

$$\mathbb{R}(b) \geq \log |\det A| + \frac{n}{2} \log \left( 1 + \frac{N(V) |\det M|^{\frac{1}{n}}}{(b - b_{\min})/n} \right). \quad (21)$$

- More generally, if  $(A, B)$  is controllable, then

$$\mathbb{R}(b) \geq \sum_{i: |\lambda_i(A)| \geq 1} \log |\lambda_i(A)|. \quad (22)$$

Applying

$$I(F_t; G_t | G^{t-1}) \leq H(F_t | G^{t-1}), \quad (23)$$

where  $H(\cdot|\cdot)$  is the conditional entropy, to (4), we see that the rate-cost function provides a lower bound on the output entropy of a quantizer achieving cost  $b$ . Furthermore, since the minimum encoded average length  $L^*(X)$  in lossless compression of an object  $X$  is bounded as [1], [2]

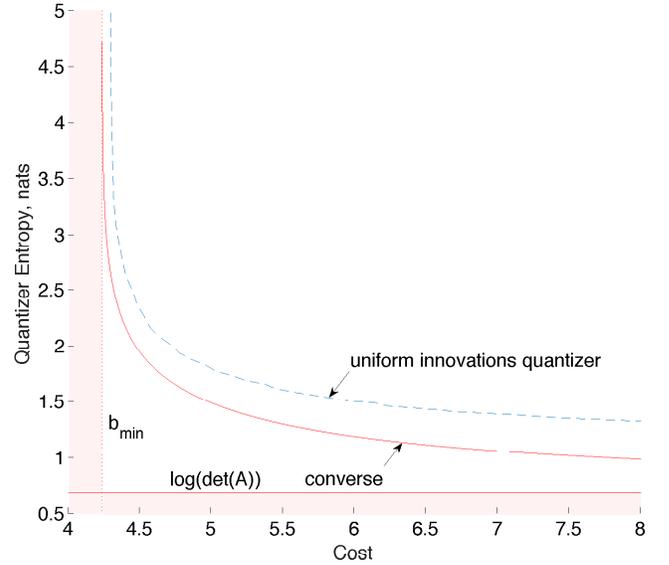
$$L^*(X) + \log(L^*(X) + 1) + \log e \geq H(X) \geq L^*(X), \quad (24)$$

the rate-cost function provides a converse (impossibility) bound on the minimum average compression rate compatible with target cost  $b$ .

Theorem 1 also gives a bound on the minimum capacity of the channel  $F_t \rightarrow G_t$  compatible with target cost  $b$  in the setting where the channel  $F_t \rightarrow G_t$  introduces random noise.

The right-hand side of (21) is a decreasing function of  $b$ , which means that the controller needs to know more information about the state of the system to attain a smaller

<sup>1</sup>All log's and exp's are common arbitrary base specifying the information units.



**Fig. 2:** The minimum quantizer entropy compatible with cost  $b$  in system (1) with parameters  $n = 1$ ,  $A = 2$ ,  $B = Q = R = 1$ ,  $V$  has Laplace distribution with variance 1.

target cost. As an important special case, consider the rate-cost tradeoff where the goal is to minimize the mean-square deviation from the desired state 0. Then,  $Q = I_n$ ,  $R = 0$ ,  $S = M = I_n$ ,  $b_{\min} = \text{Var}[V]$ , and (21) particularizes as

$$\mathbb{R}(b) \geq \log |\det A| + \frac{n}{2} \log \left( 1 + \frac{N(V)}{(b - \text{Var}[V])/n} \right). \quad (25)$$

In another important special case, namely Gaussian  $V$ , (21) particularizes as

$$\mathbb{R}(b) \geq \log |\det A| + \frac{n}{2} \log \left( 1 + \frac{|\det \Sigma_V M|^{\frac{1}{n}}}{(b - b_{\min})/n} \right). \quad (26)$$

In a pleasing confluence, for the scalar system, (26) coincides with (17).

A typical behavior of (25) is plotted in Fig. 2 as a function of target cost  $b$ . As  $b \downarrow 0$ , the required rate  $\mathbb{R}(b) \uparrow \infty$ . Conversely, as  $b \uparrow \infty$ , the rate monotonically decreases and approaches  $\log |\det A|$ . The rate-cost tradeoff provided by Theorem 1 can serve as a gauge for choosing an appropriate communication rate in order to meet the control objective. For example, in the setting of Fig. 2, decreasing the data rate below 1 nat per sample incurs a massive penalty in cost, because the rate-cost function is almost flat in that regime. On the other hand, increasing the rate from 1 to 3 nats per sample brings a lot of improvement in the attainable cost, while further increasing it beyond 3 nats results in virtually no improvement.

Also plotted in Fig. 2 is the output entropy of a variable-rate uniform scalar quantizer that takes advantage of the memory of the past only through the innovation, i.e. the difference between the controller's prediction of the state at

time  $t$  given the information the controller had at time  $t - 1$  and the true state. Its performance is strikingly close to the lower bound, being within 0.5 nat even at large  $b$ , despite the fact that quantizers in this class cannot attain the optimal cost exactly [14]. The gap further vanishes as  $b$  decreases. The gap can be further decreased for multidimensional systems by taking advantage of lattice quantization. These effects are formally captured by the achievability result we are about to present, Theorem 2.

Theorem 2 holds under the assumption that the density of the noise is sufficiently smooth. Specifically, we adopt the following notion of a regular density.

**Definition 2** (Regular density, [32]). *Let  $c_0 \geq 0$ ,  $c_1 \geq 0$ . Differentiable probability density function  $f_X$  of a random vector  $X \in \mathbb{R}^n$  is called  $(c_0, c_1)$ -regular if*

$$\|\nabla f_X(x)\| \leq (c_1\|x\| + c_0)f_X(x), \quad \forall x \in \mathbb{R}^n. \quad (27)$$

A wide class of densities satisfying smoothness condition (27) is identified in [32]. Convolution with Gaussians produces a regular density: more precisely, the density of  $B + Z$ , with  $B \perp\!\!\!\perp Z$  and  $Z \sim \mathcal{N}(0, \sigma^2 \mathbf{I})$ , is  $(\frac{4}{\sigma^2} \mathbb{E}[\|B\|], \frac{3}{\sigma^2})$ -regular.<sup>2</sup> Likewise, if the density of  $Z$  is  $(c_0, c_1)$ -regular, then that of  $B + Z$ , where  $\|B\| \leq b$  a.s.,  $B \perp\!\!\!\perp Z$  is  $(c_0 + c_1b, c_1)$ -regular.

**Theorem 2.** *Consider the fully observed linear stochastic system (1). Suppose that  $\text{rank } \mathbf{B} = n$  and  $V$  has a regular density. Then, any LQR cost  $b > b_{\min}$  is attainable by a quantizer whose output entropy is*

$$\begin{aligned} \mathbb{H}(b) \leq & \log |\det \mathbf{A}| + \frac{n}{2} \log \left( 1 + \frac{N(V) |\det \mathbf{M}|^{\frac{1}{n}}}{(b - b_{\min})/n} \right) \\ & + O_1(\log n) + O_2 \left( (b - b_{\min})^{\frac{1}{2}} \right), \end{aligned} \quad (28)$$

where  $O_1(\log n) \leq C_1 \log n$  and  $O_2(\xi) \leq C_2 \min\{\xi, c_2\}$  for some nonnegative constants  $C_1$ ,  $C_2$  and  $c_2$ .

*Proof.* An innovation-based variable-rate lattice scheme achieving (28) and its analysis is presented in the extended version [33].  $\square$

The first two terms in (28) match the first two terms in (21). The  $O_1(\log n)$  term is the penalty due to the shape of lattice quantizer cells not being exactly spherical. The  $O_2 \left( (b - b_{\min})^{\frac{1}{2}} \right)$  is the penalty due to the distribution of the innovation not being uniform. It becomes negligible for small  $b - b_{\min}$ , and the speed of that convergence depends on the smoothness parameters of the noise density. Due to (24), Theorem 2 implies the existence of a variable-rate quantizer with rate bounded by the right side of (28) that attains LQR cost  $b$ .

Theorem 1 gives a lower (converse) bound on the output entropy of quantizers that achieve the target cost  $b$ , without

<sup>2</sup>Notation  $B \perp\!\!\!\perp Z$  reads “ $B$  is independent of  $Z$ ”.

making any assumptions on the quantizer structure and permitting the use of the entire history of observation data. Theorem 2 proves that the converse can be approached by a strikingly simple quantizer coupled with a standard controller, without common randomness (dither) at the encoder and the decoder.

### III. PROOF OF THEOREM 1

We start by introducing a few crucial definitions and tools, some classical, some novel, that form the basis of our proof technique.

The traditional distortion-rate function is defined as follows.

**Definition 3** (Distortion-rate function). *Let  $X \in \mathbb{R}^n$  be a random variable. The distortion-rate function at rate  $r$  is defined as the solution to the following convex optimization problem:*

$$\mathbb{D}_r(X) \triangleq \min_{\hat{X}: I(X; \hat{X}) \leq r} \mathbb{E} \left[ (X - \hat{X})^T (X - \hat{X}) \right]. \quad (29)$$

The conditional distortion-rate function with side information  $Y$  at both the encoder and the decoder is defined as:

$$\mathbb{D}_r(X|Y) \triangleq \min_{\hat{X}: I(X; \hat{X}|Y) \leq r} \mathbb{E} \left[ (X - \hat{X})^T (X - \hat{X}) \right]. \quad (30)$$

The difference between (29) and (30) is that in the latter case, an additional information,  $Y$ , is available at the encoder and the decoder. By Jensen’s inequality,

$$\mathbb{D}_r(X) \geq \mathbb{D}_r(X|Y). \quad (31)$$

Conditional distortion-rate functions will be useful for us because both the encoder and the controller have access to the past history.

We introduce the distortion-rate function with respect to the weighted mean-square distortion as follows.

**Definition 4** (Distortion-rate function with respect to a weighted mean-square error). *Let  $X \in \mathbb{R}^n$  be a random variable, and  $\mathbf{M}$  be an  $n \times n$  positive semidefinite matrix. The distortion-rate function with respect to a weighted mean-square error at rate  $r$  is defined as the solution to the following convex optimization problem.<sup>3</sup>*

$$\mathbb{D}_{r, \mathbf{M}}(X) \triangleq \min_{\substack{F, G, \hat{X}: \\ X - F - G - \hat{X}: \\ I(F; G) \leq r}} \mathbb{E} \left[ (X - \hat{X})^T \mathbf{M} (X - \hat{X}) \right] \quad (32)$$

The corresponding conditional distortion-rate function with side information is defined as:

$$\mathbb{D}_{r, \mathbf{M}}(X|Y) \triangleq \min_{\substack{F, G, \hat{X}: \\ X - F - G - \hat{X}: \\ I(F; G|Y) \leq r}} \mathbb{E} \left[ (X - \hat{X})^T \mathbf{M} (X - \hat{X}) \right]. \quad (33)$$

<sup>3</sup>We write  $X - F - G - \hat{X}$  to designate that the random variables  $X$ ,  $F$ ,  $G$  and  $\hat{X}$  form a Markov chain in that order.

The following Proposition links the functions in Definitions 3 and 4.

**Proposition 1.** *Let  $X \in \mathbb{R}^n$  be a random variable, and let  $L$  be an  $m \times n$  matrix. The following equalities hold.*

$$\mathbb{D}_r(LX) = \mathbb{D}_{r, L^T L}(X) \quad (34)$$

$$\mathbb{D}_r(LX|Y) = \mathbb{D}_{r, L^T L}(X|Y) \quad (35)$$

*Proof.* See extended version [33].  $\square$

The following tool will be instrumental in our analysis.

**Theorem 3** (Shannon's lower bound [20]). *The distortion-rate function is bounded below as*

$$\mathbb{D}_r(X) \geq \underline{\mathbb{D}}_r(X) \quad (36)$$

$$\triangleq nN(X) \exp\left(-\frac{2r}{n}\right). \quad (37)$$

If  $X$  is a white Gaussian vector, (36) holds with equality. Thus, (36) states that the distortion-rate function of  $X$  is lower bounded by the distortion-rate function of a white Gaussian vector with differential entropy  $h(X)$ .

Although beyond Gaussian  $X$ , Shannon's lower bound is rarely attained with equality [34], it is approached at high rates; specifically, according to Linkov [35], under regularity conditions,

$$\lim_{r \rightarrow \infty} \frac{\mathbb{D}_r(X)}{\underline{\mathbb{D}}_r(X)} = 1. \quad (38)$$

If  $X \in \mathbb{R}^n$  does not have a density, then  $N(X) = 0$  and the bound in (36) is trivial.

To apply Shannon's lower bound to distortion-rate problems with a weighted mean-square error, we combine Proposition 1 and Theorem 3:

$$\mathbb{D}_{r, L^T L}(X) \geq \underline{\mathbb{D}}_{r, L^T L}(X) \quad (39)$$

$$\triangleq \underline{\mathbb{D}}_r(LX). \quad (40)$$

If  $L$  is square, the entropy power scales as

$$N(LX) = |\det L|^{\frac{2}{n}} N(X), \quad (41)$$

providing a convenient expression for  $\underline{\mathbb{D}}_r(LX)$  in terms of the entropy power of  $X$ .

Another essential component of our analysis, the entropy power inequality, was first stated by Shannon [21] and proved by Stam [22].

**Theorem 4** (Entropy power inequality [21], [22]). *If  $X$  and  $Y$  are independent, then*

$$N(X + Y) \geq N(X) + N(Y). \quad (42)$$

Equality in (42) holds if and only if  $X$  and  $Y$  are Gaussian with proportional covariance matrices.

Unlike the traditional rate-distortion theory setting in which the compressor's actions cannot affect the data, in data compression for control, the controller's action at the current time step creates the data to be compressed at the next step. The following bound to the distortion-rate function minimized over the data to be compressed will be vital in proving Theorem 1.

**Proposition 2.** *Let  $X \in \mathbb{R}^n$  be a random variable. The following inequality holds:*

$$\min_{U \in \mathbb{R}^n : I(X;U) \leq s} \mathbb{D}_r(X + U|U) \geq \mathbb{D}_{r+s}(X). \quad (43)$$

*Proof.* Write

$$\min_{U_1 : I(X;U_1) \leq s} \mathbb{D}_r(X + U_1|U_1)$$

$$= \min_{U_1 : I(X;U_1) \leq s} \min_{U_2 : I(X+U_1;U_2|U_1) \leq r} \mathbb{E} [\|X + U_1 + U_2\|^2] \quad (44)$$

$$\geq \min_{U_1, U_2 : I(X;U_1, U_2) \leq r+s} \mathbb{E} [\|X + U_1 + U_2\|^2] \quad (45)$$

$$= \mathbb{D}_{r+s}(X), \quad (46)$$

where (45) holds because  $I(X + U_1; U_2|U_1) = I(X; U_2|U_1)$  and  $I(X; U_1) + I(X; U_2|U_1) = I(X; U_1, U_2)$ .  $\square$

Using relations (34) and (35), we note that (43) implies

$$\min_{U \in \mathbb{R}^n : I(X;U) \leq s} \mathbb{D}_{r, M}(X + U|U) \geq \mathbb{D}_{r+s, M}(X). \quad (47)$$

*Remark 1.* Since Shannon's lower bound is equal to the distortion-rate function of a white Gaussian vector with the same differential entropy as the original vector, the relation shown in Proposition 2 also holds between the corresponding Shannon lower bounds:

$$\min_{U \in \mathbb{R}^n : I(X;U) \leq s} \underline{\mathbb{D}}_{r, M}(X + U|U) \geq \underline{\mathbb{D}}_{r+s, M}(X). \quad (48)$$

We are now fully equipped to prove Theorem 1.

*Proof of Theorem 1.* Here, we prove (21). We refer the reader to the extended version [33] for the proof of (22). Without loss of generality, assume that  $\mathbb{E}[X_0] = \mathbb{E}[V] = 0$ . We consider the finite horizon problem in which the system operates for  $t$  time steps and the goal is to minimize the quadratic cost function in (2). To find a lower bound to the minimum attainable cost  $\mathbb{B}_t(r)$ , we apply the dynamic programming principle.

Fix  $t \geq 1$ . For  $i \leq t$ , denote for brevity

$$b_{i+1} \triangleq \mathbb{E} \left[ \sum_{j=0}^i (X_j^T Q X_j + U_j^T R U_j) \right] + \mathbb{E} [X_{i+1}^T S_{i+1} X_{i+1}], \quad (49)$$

$$L_i \triangleq (R + B^T S_{i+1} B)^{-1} B^T S_{i+1}, \quad (50)$$

$$M_i \triangleq S_{i+1} B (R + B^T S_{i+1} B)^{-1} B^T S_{i+1} \quad (51)$$

$$= L_i^T (R + B^T S_{i+1} B) L_i, \quad (52)$$

where  $S_i$ ,  $i \leq t$  is the solution to the discrete-time Ricatti recursion,

$$S_i = Q + A^T (S_{i+1} - M_i) A, \quad (53)$$

and  $S_{t+1}$  is that in (2).

Suppose that the controls  $U_1, \dots, U_{i-1}$  have already been chosen, producing the random variable  $X_i$  at the output of the system at time  $i$ . Denote by  $J_i(X_i)$  the minimum (over  $U_i, \dots, U_t$ ) expected cost at time  $t$ , given a fixed choice of  $U_1, \dots, U_{i-1}$ .<sup>4</sup>

The expected cost at time  $t+1$  for given  $U_1, \dots, U_t$  is

$$J_{t+1}(X_{t+1}) \triangleq b_{t+1}. \quad (54)$$

To find  $J_t(X_t)$ , we optimize  $J_{t+1}(X_{t+1})$  over  $U_t$ :

$$J_t(X_t) \triangleq \min_{U_t \in \mathcal{G}_t} J_{t+1}(AX_t + BU_t + V_t) \quad (55)$$

$$= \mathbb{E} \left[ \sum_{j=0}^{t-1} (X_j^T Q X_j + U_j^T R U_j) \right] + \min_{U_t \in \mathcal{G}_t} \mathbb{E}[X_t^T Q X_t + U_t^T R U_t + (AX_t + BU_t + V_t)^T S_{t+1} (AX_t + BU_t + V_t)] \\ = b_t + \mathbb{E} [V_t^T S_{t+1} V_t] \quad (56)$$

$$+ \min_{U_t \in \mathcal{G}_t} \mathbb{E}[(L_t A X_t + U_t)^T (R + B^T S_{t+1} B) (L_t A X_t + U_t)] \\ \geq b_t + \text{tr}(\Sigma_V S_{t+1}) + \mathbb{D}_{r, R+B^T S_{t+1} B}(L_t A X_t | U^{t-1}), \quad (57)$$

where

- (56) is obtained by completing the squares in the standard way, see e.g. [12].
- (57) holds because  $\mathbb{E} [V_t^T S_{t+1} V_t] = \text{tr}(\Sigma_V S_{t+1})$ , and because by Proposition 1, (31) and Shannon's lower bound (Theorem 3), the minimum with respect to  $U_t$  in the left side of (57) is lower bounded by

$$\geq \mathbb{D}_{r, R+B^T S_{t+1} B}(L_t A X_t | U^{t-1}). \quad (58)$$

Note that (58) holds regardless of whether the entire history of  $U^{t-1}$  is available at both the encoder and the decoder or at the decoder only.

Continuing the recursion further results in:

$$J_{t-1}(X_{t-1}) \triangleq \min_{U_{t-1} \in \mathcal{G}_{t-1}} J_t(AX_{t-1} + BU_{t-1} + V_{t-1}) \quad (59)$$

$$\geq b_{t-1} + (\text{tr}(\Sigma_V S_t) + \text{tr}(\Sigma_V S_{t+1})) \\ + \mathbb{D}_{r, R+B^T S_t B}(L_{t-1} A X_{t-1} | U^{t-2}) \\ + \min_{U_{t-1} \in \mathcal{G}_{t-1}} \mathbb{D}_{r, R+B^T S_{t+1} B}(L_t A X_t | U^{t-1}) \quad (60)$$

$$\geq b_{t-1} + (\text{tr}(\Sigma_V S_t) + \text{tr}(\Sigma_V S_{t+1})) \\ + \mathbb{D}_{r, R+B^T S_t B}(L_{t-1} A X_{t-1} | U^{t-2}) \\ + \mathbb{D}_{2r, R+B^T S_{t+1} B}(L_t A^2 X_{t-1} | U^{t-2}) \\ + \mathbb{D}_{r, R+B^T S_{t+1} B}(L_t A V) \quad (61)$$

<sup>4</sup>As common in information theory, we will abuse the notation slightly and write  $J_i(X_i)$  even though it is a function of the distribution of  $X_i$  only and not of the particular realization of  $X_i$ .

where (60) repeats (57), and (61) is obtained using the entropy power inequality, (48) and (58) as follows:

$$\mathbb{D}_{r, R+B^T S_{t+1} B}(L_t A (AX_{t-1} + BU_{t-1} + V_{t-1}) | U^{t-1}) \quad (62)$$

$$\geq \mathbb{D}_{r, R+B^T S_{t+1} B}(L_t A (X_{t-1} + BU_{t-1}) | U^{t-1}) \\ + \mathbb{D}_{r, R+B^T S_{t+1} B}(L_t A V) \quad (63)$$

$$\geq \mathbb{D}_{2r, R+B^T S_{t+1} B}(L_t A^2 X_{t-1} | U^{t-2}) + \mathbb{D}_{r, R+B^T S_{t+1} B}(L_t A V) \quad (64)$$

Backtracking all the way to time 0, we accumulate

$$J_0(X_0) \triangleq \min_{U_0 \in \mathcal{G}_0} J_1(AX_0 + BU_0 + V_0) \quad (65)$$

$$\geq \mathbb{E} [X_0^T S_0 X_0] + \sum_{i=0}^t \text{tr}(\Sigma_V S_{i+1}) \\ + \sum_{i=0}^t \mathbb{D}_{(i+1)r, R+B^T S_{i+1} B}(L_i A^{i+1} X_0) \\ + \sum_{i=0}^t \sum_{j=1}^i \mathbb{D}_{j r, R+B^T S_{i+1} B}(L_i A^j V). \quad (66)$$

Now, we drop the terms containing  $X_0$  and apply the scaling property (41) to continue (66) as follows.

$$J_0(X_0) \geq \mathbb{E} [X_0^T S_0 X_0] + \sum_{i=0}^t \text{tr}(\Sigma_V S_{i+1}) \quad (67)$$

$$+ n \sum_{i=0}^t (\det M_i)^{\frac{1}{n}} N(V) \sum_{j=1}^i |\det A|^{\frac{2j}{n}} \exp\left(-\frac{2jr}{n}\right).$$

Denote for brevity  $a = |\det A|$ . Note that

$$\sum_{j=1}^i a^{\frac{2j}{n}} \exp\left(-\frac{2jr}{n}\right) = \frac{1 - \exp\left(-\frac{2i}{n}(r - \log a)\right)}{\exp\left(\frac{2}{n}(r - \log a)\right) - 1}. \quad (68)$$

If  $r \leq \log a$ , the contribution of the sum containing  $N(V)$  in (67) grows faster than  $t$ , and so

$$\limsup_{t \rightarrow \infty} \mathbb{B}_t(r) = +\infty. \quad (69)$$

Consider now the case  $r > \log a$ . Taking  $t \rightarrow \infty$ , we conclude

$$\limsup_{t \rightarrow \infty} \mathbb{B}_t(r) \geq \lim_{t \rightarrow \infty} \frac{1}{t+1} \sum_{i=0}^t \text{tr}(\Sigma_V S_{i+1}) \\ + \lim_{t \rightarrow \infty} \frac{nN(V)}{t+1} \sum_{i=0}^t (\det M_i)^{\frac{1}{n}} \frac{1 - \exp\left(-\frac{2i}{n}(r - \log a)\right)}{\exp\left(\frac{2}{n}(r - \log a)\right) - 1} \quad (70)$$

$$\geq \text{tr}(\Sigma_V S) + \frac{nN(V) |\det M|^{\frac{1}{n}}}{\exp\left(\frac{2}{n}(r - \log a)\right) - 1}, \quad (71)$$

where (71) is by Fatou's lemma.

Finally, since  $S > 0$  (e.g. [12]) and  $\text{rank } B = n$  by the assumption, it follows that  $\det M > 0$ , and (71) is equivalent to (21).  $\square$

#### IV. CONCLUSION

We studied the fundamental tradeoff between the communication requirements and the attainable quadratic cost in the fully observed linear stochastic control system. We introduced the rate-cost function in Definition 1, and showed a sharp lower bound to it in Theorem 1. The proof uses backwards induction and invokes Shannon's lower bound and the entropy power inequality to lower bound the cost for any admissible control sequence. Theorem 2 shows that a variable-rate lattice-based scheme in which only the quantized value of the innovation is transmitted approaches the converse result if the target cost is not too high and the system dimension is not too small.

It remains an interesting open question whether the converse bound in Theorem 1 can be approached by fixed-rate quantization, or over noisy channels. It will be also interesting to see whether using non-lattice quantizers can help to narrow down the gap in Fig. 2.

#### V. ACKNOWLEDGEMENT

The authors acknowledge many stimulating discussions with Dr. Anatoly Khina and his helpful comments on the earlier versions of the manuscript. The authors are also grateful to Ayush Pandey, who with incessant enthusiasm generated the plot in Fig. 2.

The work of Victoria Kostina was supported in part by the National Science Foundation (NSF) under Grant CCF-1566567. The work of Babak Hassibi was supported in part by the National Science Foundation under grants CNS-0932428, CCF-1018927, CCF-1423663 and CCF-1409204, by a grant from Qualcomm Inc., by NASA's Jet Propulsion Laboratory through the President and Director's Fund, and by King Abdullah University of Science and Technology.

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