Stabilizing Dynamical Systems with Fixed-Rate Feedback using Constrained Quantizers

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The model

- A linear dynamical system

\[ X_{t+1} = AX_t + BU_t + Z_t \]

- In practical scenarios, the observer and controller are not co-located
- The traditional observer and controller also serve as encoder-decoder
- The mappings are online (causal)
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Motivation and related works

Networked control settings:
- Wearable devices
- Remote sensors
- Drones
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Control-Communication hybrid systems:

- Stabilizing dynamical systems using communication
  [Tatikonda, Sahai, Mitter 04], [Elia 04], [Borkar 97], [Silva, Derpich, Ostergaard, Encina 16], [Nair, Evans 04]

- Tradeoff between LQG cost and communication:
  [Tanaka, Kim, Parrilo, Mitter 17], [Fox, Tishby 16], [Khina, Nakahira, Su, Yildiz, Hassibi 18], [Tanaka, Esfahani, Mitter 17], [Sabga, Tian, Kostina, Hassibi 20]
The setting

\[ S_t \in [1 : |S|] \]

\[ X_{t+1} = aX_t - U_t + Z_t \]

\[ X_t \]

\[ U_t \]

\[ \text{Obs.} \]

\[ \text{Cont.} \]

- The noise \( Z_t \) has an \( \alpha \)-bounded moment

\[ E[|Z_t|^\alpha] < \infty. \]

- The observer mapping: \( X_1, \ldots, X_t \rightarrow S_t, \quad S_t \in [1 : |S|] \)
- The controller mapping: \( S_1, \ldots, S_t \rightarrow U_t \)
- The objective: to stabilize the system

- A dynamical system is \( \beta \)-stable if there exists a sequence of observer-controller mappings such that

\[ \limsup_{t \to \infty} E[|X_t|^\beta] < \infty. \]
The converse

Theorem (Nair, Evans 04)

Any scheme which stabilizes a dynamical system with Gaussian noise satisfies

\[ |S| > a. \]

- Very simple converse noise for bounded moments [Kostina et al. 18]
- In [Nair et al. 04], the rate is achievable with variable-rate communication

In the fixed-rate setting, \(|S|\) is an integer, implying

\[ |S^*| \geq \lfloor a \rfloor + 1 \]
Zoom-in/zoom-out schemes

- Simple case - no noise:

\[ X_{t+1} = aX_t - U_t \]

1. An interval \([-C_t, C_t]\) is known to the Obs. and the Cont.
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2. The Observer transmits the state quantization

3. The Cont. applies \(U_t = a\hat{x}_t\) (\(\hat{x}_t\) is the midpoint of the cell)
   - The new state:
   \[
   |X_{t+1}| = a|x_t - \hat{x}_t| \leq a\frac{C_t}{|S|}
   \]
Zoom-in/zoom-out schemes

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   - The new state:

   \[ |X_{t+1}| = a|x_t - \hat{x}_t| \leq a \frac{C_t}{|S|} \]

4. If \( a < |S| \), the interval can be zoomed-in as \( C_{t+1} = rC_t \)

   where \( r \) satisfies \( \frac{a}{|S|} \leq r < 1 \)
Zoom-in/zoom-out schemes

- With noise:
  \[ X_{t+1} = aX_t - U_t + Z_t \]

- If the noise has bounded support \(|Z_t| \leq \Delta|\):
  \[ |X_{t+1}| = a|x_t - \hat{x}_t| + |Z_t| \leq a\frac{c}{|S|} + \Delta \]
  and zoom-in (plus an additive term) is sufficient:
  \[ C_{t+1} = rC_t + \Delta \]

- For noise with unbounded support (e.g., Gaussian), the state may "escape" the interval
  - Thus, we may need to zoom-out as \( C_{t+1} = PC_t \) with \( P > 1 \)
  - The transition to zoom-out should be communicated!
How to communicate the transition

- The minimal rate to achieve is
  \[ |S^*| = \lfloor a \rfloor + 1 \]

- If a symbol is dedicated for communicating transitions,
  \[ |S| = \lfloor a \rfloor + 2 \]

  is achievable (Yuksel 10)

- Optimal scheme with non-explicit coding parameters
  (Kostina, Peres, Ranade, Sellke 18)

- Our main idea is to encode the transition over several times using constrained quantizer

- Our main contribution is the constrained quantizer:  
  - communicates the transitions with the optimal rate  
  - Precise analysis leads to explicit scheme
The constrained quantizer

- Constrained coding (avoiding patterns) is popular in storage media
- For instance, the \((0, l - 1)\)-RLL constrained quantizer \((|S| = 2\) and \(l = 2, 3\))

\[
\begin{align*}
00 & \quad 01 & \quad 10 & \quad 11 & \quad 00 \\
-c & \quad & \quad & \quad & \quad c
\end{align*}
\]

\[
\begin{align*}
000 & \quad 001 & \quad 010 & \quad 011 & \quad 100 & \quad 101 & \quad 110 & \quad 111 & \quad 000 \\
-c & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad c
\end{align*}
\]

- The 00.. sequence is for transition to zoom-out
- We can always choose \(l\) such that there is a zoom-in, i.e.,

\[
\frac{a^l}{([a] + 1)^l - 1} < 1
\]
The algorithm

Inputs: $c_0, l, r, \Delta, P$

$[x, c] \leftarrow \text{Zoom-Out}(x, c_0, P)$

procedure

$s^l \leftarrow Q_C(x, c, l)$

$S_i \leftarrow s_i$, for $i = 1, \ldots, l$

$U_i \leftarrow 0$, for $i = 1, \ldots, l - 1$

if $s^l \neq 0^l$ then

$U_l \leftarrow U(S^l)$

$c \leftarrow r \cdot c + \Delta$

else

$[x, c] \leftarrow \text{Zoom-Out}(x, c, P)$

end if

end procedure
Main result

Theorem (Algorithm optimality)

Any dynamical system with $\mathbb{E}[|X_0|^\alpha] \leq \rho_\alpha$ is $\beta$-stable, for $\beta < \alpha$, using the algorithm with $|S| = \lfloor a \rfloor + 1$ if

$$1 > r \geq \frac{a^l}{(\lfloor a \rfloor + 1)^l - 1}$$

$$P > a^{\lfloor a \rfloor (\alpha - \beta)}$$

$$\Delta^\alpha > \left( \frac{\ln(P^\beta)}{1 - \frac{a^\alpha}{P^\alpha - \beta}} \frac{a^{\alpha l} \rho_\alpha}{(1 - a)^\alpha} \right)^{2^\beta - 1}.$$

- We can always choose $l$ s.t. $\frac{a^l}{(\lfloor a \rfloor + 1)^l - 1} < 1$
Proof idea

- Analysis of the states by the end of procedures
- Each procedure has a random duration
- The explicit parameters are due to the following upper bound

Lemma (Bounded sums-moment)

Let $Z_i$ be random variables with $E[|Z_i|^\alpha] \leq \rho_\alpha$ and $a > 1$. Then, for any $\beta \leq \alpha$,

$$E\left[\left| \sum_{j=0}^{i} a^{-j} Z_j \right|^\beta \right] \leq \rho_\alpha \left( \frac{1 - a^{-i}}{1 - a^{-1}} \right)^\beta.$$
We assume that

\[ A = \bigoplus_{i=1}^{\Lambda} J_i, \]

where \( J_i \) is a Jordan block with dimension \( m_i \)

The pair \( (A, B) \) is controllable

Define \( a = \prod_{i=1}^{d} \max\{1, |\lambda_i|\} \). The converse gives

\[ |S^*| \geq \lfloor a \rfloor + 1 \]
Solution to MIMO dynamical systems

- We use time-sharing between the Jordan blocks
- Each Jordan block will use our SISO algorithm for $l_i$ transmissions (extension to Jordan blocks is trivial)
- One of the Jordan block will use the constrained quantizer

**Lemma (Feasible time-sharing solution)**

For any $\{\lambda_1, \ldots, \lambda_\Lambda\}$ with $a = \prod_{i=1}^{\Lambda} |\lambda_i|$, there exists a sequence $\{l_i\}_{i=1}^{\Lambda}$ with $L = \sum_{i=1}^{\Lambda} l_i$ such that

$$|\lambda_i|^L \leq (\lfloor a \rfloor + 1) \frac{l_i}{m_i}$$

for all $i = 1, \ldots, \Lambda$, and

$$|\lambda_i|^L \leq (\lfloor a \rfloor + 1) \frac{l_i}{m_i} - 1$$

for some $i$. 

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Ongoing research:
Construction of an algorithm for systems with multiple (partial) observers
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Thank you very much!